

Shape of the inflaton potential and the efficiency of the universe heating

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It is shown that the efficiency of the universe heating by an inflaton field depends not only on the possible presence of parametric resonance in the production of scalar particles but also strongly depends on the character of the inflaton approach to its mechanical equilibrium point. In particular, when the inflaton oscillations deviate from pure harmonic ones toward a succession of step functions, the production probability rises by several orders of magnitude. This in turn leads to a much higher temperature of the universe after the inflaton decay, in comparison to the harmonic case. An example of the inflaton potential is presented which creates a proper modification of the evolution of the inflaton toward equilibrium and does not destroy the nice features of inflation.

I. INTRODUCTION

Cosmological inflation consisted, roughly speaking, of two epochs. The first one was a quasi-exponential expansion, when the Hubble parameter, H , slowly changed with time and the universe expanded by a huge factor, e^N , where

$$N = \int H dt \gg 1. \quad (1.1)$$

During this period the Hubble parameter exceeded the inflaton mass or, rather, the square of the Hubble parameter was larger than the second derivative of the inflaton potential:

$$H^2 > \left| \frac{d^2 U(\phi)}{d\phi^2} \right| \equiv |U''(\phi)|, \quad (1.2)$$

where $U(\phi)$ is the potential of the inflaton field, ϕ . Due to the large Hubble friction (see Eq. (3.3)) during this epoch the field ϕ remained almost constant, slowly moving in the direction of the "force", $-U'(\phi)$.

The second stage began when H^2 dropped below $|U''(\phi)|$ and continued till the inflaton field reached the equilibrium value where $U'(\phi_{eq}) = 0$. It is usually assumed that $\phi_{eq} = 0$ and $U(\phi_{eq}) = 0$. The last condition is imposed to avoid a nonzero vacuum energy. During this period ϕ oscillated around ϕ_{eq} , producing elementary particles, mostly with masses smaller than the frequency of the inflaton oscillations. This was a relatively short period which may be called a big bang, when the initial dark vacuum-like state exploded, creating hot primeval cosmological plasma.

The process of the universe heating was first studied in Refs. [1–3] within the framework of perturbation theory. A non-perturbative approach was pioneered in Refs. [4–6], where the possibility of excitation of parametric resonance which might grossly enhance the particle (boson) production rate was mentioned. In the model of Ref. [4] parametric resonance could not be effectively induced because of the redshift and scattering of the produced particles which were dragged out of the resonance zone and the main attention in this work was set on non-perturbative production of fermions. However, the resonance may be effective if it is sufficiently wide. In this case the particle production rate can be strongly enhanced [5–8].

As is well known, parametric resonance exists only in the process of the boson production. In quantum language it can be understood as Bose amplification of particle production due to the presence of identical bosons in the final state; it is the same phenomenon as the induced radiation in laser. For bosons there could be another phenomenon leading to very fast and strong excitation of the bosonic field coupled to inflaton, if the effective mass squared of such field became negative (a tachyonic situation) [9–11]. It happens for sufficiently large and negative product

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$g\phi$; see below Eqs. (2.1), (2.2). This is similar to the Higgs-like effect, when the vacuum state becomes unstable. However, in contrast to the Higgs phenomenon, this took place only during a negative half-wave of the inflaton oscillations.

Both phenomena are absent in the case of fermion production. The imaginary mass of the fermions breaks the hermiticity of the Lagrangian, so tachyons must be absent. As for parametric resonance, it is not present in the fermionic equations of motion. The latter property is attributed to the Fermi exclusion principle. A non-perturbative study of the fermion production shows that the production probability sharply grows when the effective mass of the fermions crosses zero [4, 12].

The resonance amplification of the particle production by the inflaton exists not only in the canonical case of harmonic inflaton oscillations, i.e. for the potential $m^2\phi^2/2$, but for a very large class of inflaton potentials. The explicit magnitude of the production probability depends, of course, upon the shape of the potential, but there is no big difference between different power law potentials, $\sim \phi^n$. However, as we show in this work, the production rate could be drastically enhanced for some special forms of the potential, if the potential noticeably deviates from a simple power law.

There are several more phenomena which might have an impact on the particle production probability. In Refs. [13–15] the effects of quantum or thermal noise on the inflaton evolution have been studied. It was argued by these two groups that the resonance is not destroyed by noise. If the noise is not correlated temporally it would lead on the average to an increase in the rate of particle production [14, 15]. This result is valid both for homogeneous and inhomogeneous noise. However, the cosmological expansion was neglected in Refs. [14, 15] and it possibly means that the resonance, to survive in realistic cosmological situations, should be sufficiently wide.

The efficiency of the production depends also on the model of inflation. In particular, in the case of multifield inflation the canonical parametric resonance is suppressed [16–18]. However, efficient (pre)heating is possible via tachyonic effects. To this end a trilinear coupling of the inflaton to light scalars is necessary. If the tachyonic mechanism is not operative, the old perturbative approach [1–3] would be applicable.

A new mechanism of enhanced preheating after multifield inflation has been found in the recent paper [19], due to the presence of extra produced species which became light in the course of the multifield inflaton evolution.

The efficiency of different earlier scenarios of the cosmological heating after inflation is discussed in a large number of review papers, e.g. [20], where one can find an extensive list of literature. More recent development is described in reviews [21, 22].

In this paper we study a parametric resonance excitation for different forms of the inflaton potential, $U(\phi)$. A new effect is found: for some non-harmonic potentials of a single field inflation parametric resonance (not tachyonic) is excited considerably stronger than in the case of simple power law potentials, both in flat space-time and in cosmology. Correspondingly, the cosmological particle production by the end of inflation would be much more efficient, and the temperature of the created plasma would become noticeably higher. In Sect. II we consider this problem in flat space-time to get a feeling for a proper choice of the inflaton potential that could generate the signal $\phi(t)$ most efficient for the particle production. Consideration of the flat space-time example clearly demonstrates the essence of the effect which is not obscured by the cosmological expansion. In Sect. III we study the evolution of the inflaton field in cosmological background for different potentials $U(\phi)$. Based on the example considered in Sect. II, we found a potential for which the inflaton induces parametric resonance much more efficiently than in the purely harmonic case, or other simple power law potentials. We also comment there on the properties of inflationary cosmology with such modified inflaton potentials. In this section the scalar particle production rate by such "un-harmonic" inflaton is calculated. The results are compared to the particle production rate for the usual harmonic oscillations of the inflaton. Section IV is dedicated to an estimate of the effects of back reaction of the particle production on the inflaton evolution. In Sect. V we draw conclusions.

We have chosen the sign and the amplitude of the initial inflaton field to avoid or to suppress tachyonic amplification of the produced field χ .

II. PARAMETRIC RESONANCE IN FLAT SPACE-TIME

Let us consider at first an excitation of parametric resonance in the classical situation, when the space-time curvature is not essential and the Fourier amplitude of the would-be resonating scalar field χ satisfies the equation of motion

$$\ddot{\chi} + (m_\chi^2 + k^2 + g\phi)\chi = 0, \quad (2.1)$$

where m_χ is the mass of χ , k is its momentum, and g is the coupling constant between χ and another scalar field ϕ with the interaction

$$L_{int} = -\frac{1}{2}g\phi\chi^2. \quad (2.2)$$

The classical field ϕ is supposed to be homogeneous, $\phi = \phi(t)$ and to satisfy the equation of motion:

$$\ddot{\phi} + U'(\phi) = -\frac{1}{2}g\chi^2. \quad (2.3)$$

Below we mostly neglect the effects of the r.h.s. term in this equation. Its impact on the inflaton evolution is relatively weak. It introduces the back reaction of the particle production and is discussed in Sect. IV.

Our task here is to determine $U(\phi)$, so that the parametric resonance for χ would be most efficiently excited. An optimal meander form of the inflaton field ϕ is suggested by the phase parameter approach in the theory of parametric resonance [23]. Here we demonstrate that just a slight shift from the standard Mathieu model toward an optimal inflaton potential leads to a drastic increase of the particle production rate. We compare the modified results with the standard case when the potential of ϕ is quadratic, $U(\phi) = m^2\phi^2/2$, and therefore Eq. (2.3) with zero r.h.s. has a solution $\phi(t) = \phi_0 \cos(mt + \theta)$, where the amplitude ϕ_0 and the phase θ can be found from the initial conditions. One can choose the moment $t = 0$ in such a way that $\theta = 0$, i.e. $\phi(t) = \phi_0 \cos mt$. Substituting this expression into Eq. (2.1), we come to the well-known Mathieu equation:

$$\ddot{\chi} + \omega_0^2(1 + h \cos mt)\chi = 0, \quad (2.4)$$

where $\omega_0^2 = m_\chi^2 + k^2$ and $h = g\phi_0/\omega_0^2$. When $h \ll 1$ and the value of m is close to $2\omega_0/n$ (where n is an integer), Eq. (2.4) describes a parametric resonance, i.e. field χ oscillates with an exponentially growing amplitude. For $h \ll 1$ the solution of Eq. (2.4) can be represented as a product of a slowly (but exponentially) rising amplitude by a quickly oscillating function with frequency ω_0 :

$$\chi = \chi_0(t) \cos(\omega_0 t + \alpha). \quad (2.5)$$

The amplitude χ_0 satisfies the equation

$$-\ddot{\chi}_0 \cos(\omega_0 t + \alpha) + 2\omega_0 \dot{\chi}_0 \sin(\omega_0 t + \alpha) = h\omega_0^2 \chi_0 \cos mt \cos(\omega_0 t + \alpha). \quad (2.6)$$

Let us multiply Eq. (2.6) by $\sin(\omega_0 t + \alpha)$ and average over the period of oscillations. The right hand side would not vanish on the average, if $m = 2\omega_0$. In this case χ_0 would exponentially rise if $\alpha = \pi/4$:

$$\chi_0 \sim \exp\left(\frac{1}{4}h\omega_0 t\right). \quad (2.7)$$

In this way we recovered the standard results of the parametric resonance theory.

The rise of the amplitude of χ is determined by the integral

$$\frac{1}{2}\dot{\chi}^2 + \frac{1}{2}\omega_0^2\chi^2 = -g \int dt \chi \dot{\chi} \phi, \quad (2.8)$$

as one can see from Eq. (2.1). It can be shown that the maximum rate of the rise is achieved when ϕ is a quarter-period meander function (an oscillating succession of the step-functions with proper step duration); see Ref. [23].

Let us note that the behavior of the solution for χ would dramatically change with rising h . For a large h the eigenfrequency squared of χ noticeably changes with time. It may approach zero and, if $|h| > 1$, it would even become negative for a while; see Eq. (2.4). In the latter (tachyonic) case χ would rise much faster than in the case of classical parametric resonance. We postpone the study of tachyonic case for a future work, while here we confine ourselves to a non-tachyonic situation.

To demonstrate an increase of the excitation rate for an anharmonic oscillation we choose, as a toy model, the potential for the would-be inflaton field ϕ satisfying the following conditions: at small ϕ it approaches the usual harmonic potential, $U \rightarrow m^2\phi^2/2$, while for $\phi \rightarrow \infty$ it tends to a constant value. It is intuitively clear that in such a potential field ϕ would live for a long time in the flat part of the potential and quickly change sign near $\phi = 0$. This behavior can be rather close to a periodic succession of the step-functions which we mentioned above. As an example of such a potential we take

$$U(\phi) = \frac{1}{2}m^2\phi^2 \cdot \frac{1 + \lambda_0(\phi/m_{Pl})^2}{1 + \lambda_2(\phi/m_{Pl})^2 + \lambda_4(\phi/m_{Pl})^4}, \quad (2.9)$$

where $m, \lambda_0, \lambda_2, \lambda_4$ are some constant parameters, with m having dimension of mass, the λ_j being dimensionless. Here m_{Pl} is the Planck mass. Observational data on the density perturbations induced by the inflaton demand $m \simeq 10^{-6}m_{Pl}$ in the model with the inflaton potential $U(\phi) = m^2\phi^2/2$, so we use $m = 10^{-6}m_{Pl}$ as a reference value throughout the paper. Since the potential of our toy model is different from the harmonic one, the inflationary density perturbations could also be different. With the particular choice of parameters λ_j , which is presented below,

the density perturbations require a somewhat smaller m , approximately by an order of magnitude. However, our aim here is not the construction of a realistic inflationary model, but the demonstration of a new phenomenon of more efficient excitation of the parametric resonance for some forms of inflaton oscillations. To this end we take $\lambda_0 = 85$, $\lambda_2 = 4$, $\lambda_4 = 1$. The plots of the potentials are shown in Fig. 1. Here and below the red (or dashed) curves are for the quadratic potential $U(\phi) = m^2\phi^2/2$ and the blue ones are for potential (2.9). For our choice of λ parameters the plots cross each other at the points $\phi = 0, \pm 9 m_{Pl}$.

We did not look for a theoretical justification for the chosen form of the potential (2.9), bearing in mind the large freedom for possible forms of the scalar field potential at large mass/energy scale, but it is noteworthy that a similar type of the potential which is quadratic near the minimum and is flatter away from the minimum was studied in Refs. [24, 25]. Such potentials were in turn derived in Refs. [26–31]. Quoting Ref. [25]: "This choice was motivated by monodromy and supergravity models of inflation [26–30] and a recent model of axion quintessence [31]".

Figure 1 for the potential (2.9) has a shape which is quite close to that depicted in Fig. 1 from Ref. [25]. The properties of the inflationary model, which might be realized with the inflaton potentials of the type (2.9) can be understood from the results of Refs. [24, 25].

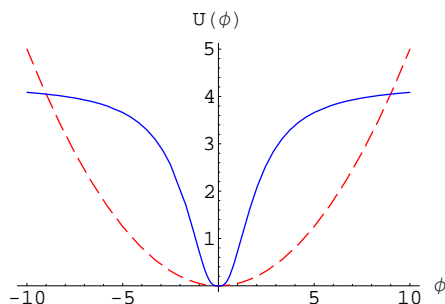


FIG. 1: The potential of the inflaton $U(\phi)$. The red dashed line is the quadratic potential $U(\phi) = m^2\phi^2/2$, and the blue line is the potential (2.9). The parameters are $m = 10^{-6}m_{Pl}$, $\lambda_0 = 85$, $\lambda_2 = 4$, $\lambda_4 = 1$. The field ϕ is measured in units of m_{Pl} and $U(\phi)$ is measured in units of $10^{-11}m_{Pl}^4$.

We solved Eq. (2.3) with zero r.h.s. and with the potential (2.9) numerically, using Mathematica (here as well as in the rest of the paper), and compared this solution, $\phi_U(t)$, with the harmonic solution, $\phi_h(t) = \phi_0 \cos mt$. The results are presented in Fig. 2 for the initial conditions $\phi(0) = 4.64 m_{Pl}$, $\dot{\phi}(0) = 0$. The value of the amplitude $\phi_0 = 4.64$ is chosen because in this case the frequencies of $\phi_U(t)$ and $\phi_h(t)$ are approximately equal.

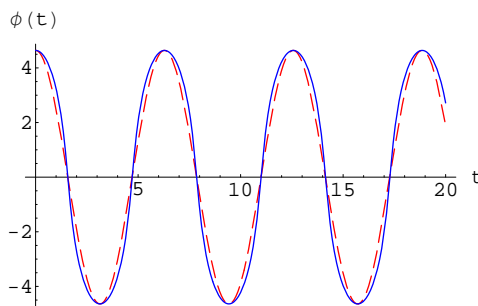


FIG. 2: Comparison of the numerical solution $\phi_U(t)$ of Eq. (2.3) with the potential (2.9) (blue line) and the harmonic solution $\phi_h(t) = \phi_0 \cos mt$ with $\phi_0 = 4.64$ and $m = 10^{-6}m_{Pl}$ (red dashed line). The time t is measured in units of m^{-1} and the field ϕ is measured in units of m_{Pl} .

For the chosen shape of the potential (2.9), the function $\phi_U(t)$ differs from a cosine toward the step function. Therefore, one can expect that parametric resonance would be excited stronger for the potential (2.9) than for the quadratic one.

Now we can numerically solve Eq. (2.1) with the computed $\phi(t)$ and with the chosen values for this example: $m_\chi = 0$, $k = 1$, $g = 5 \cdot 10^{-8}$ (in units of m), and the initial conditions $\chi(0) = 1/\sqrt{2\omega(0)} \approx 0.67$, $\dot{\chi}(0) = \sqrt{\omega(0)/2} \approx 0.745$, where $\omega^2(0) = k^2 + g\phi(0)$. These initial conditions correspond to the solution of Eq. (2.1) with constant ω : $\chi(t) = e^{-i\omega t}/\sqrt{2\omega V}$, where $V = 1$ (in units of m^{-3}) is the volume. As a result we obtain function $\chi(t)$ shown in Fig. 3. As expected, the amplitude of oscillations of χ increases faster in the case of the potential (2.9); e.g. at $t = 450 m^{-1}$ the amplitude ratio is approximately 2.

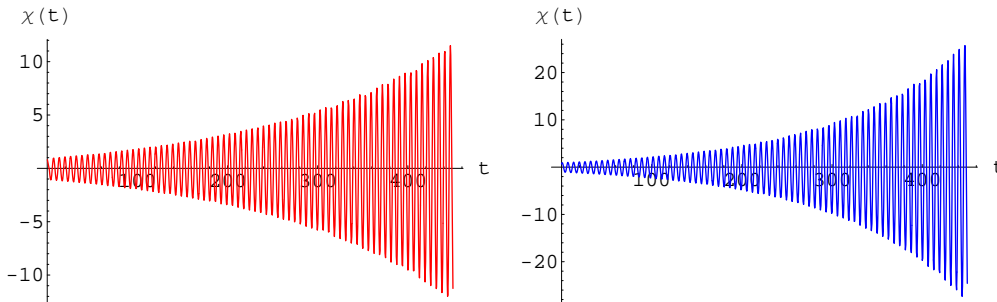


FIG. 3: Oscillations of the field $\chi(t)$. *The left panel* corresponds to harmonic potential of ϕ and *the right panel* corresponds to the potential (2.9). The time t is measured in units of m^{-1}

Since we are going to apply the results for the calculation of the particle production rate at the end of cosmological inflation, it would be appropriate to present the number density of the produced χ -particles, $n_k(t)$, which is defined as

$$n_k(t) = \left(\frac{\dot{\chi}^2}{2} + \frac{\omega_k^2 \chi^2}{2} \right) \frac{1}{\omega_k}, \quad (2.10)$$

where the expression in the brackets is the energy of the mode with momentum k , and $\omega_k = \sqrt{k^2 + g\phi}$ is the energy of one χ -particle.

The number densities of the produced χ -particles for the harmonic ϕ and slightly step-like one are presented in Fig. 4 by red and blue curves, respectively.

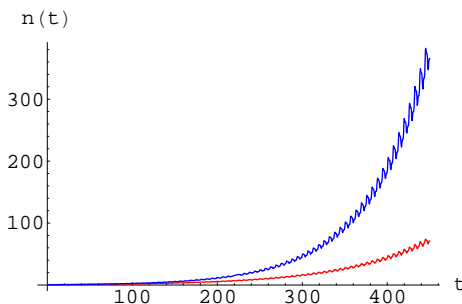


FIG. 4: The number density of produced χ -particles $n_k(t)$. The time t is measured in units of m^{-1}

Despite a decrease of particle number densities in some short time intervals, there is an overall exponential rise, which goes roughly as χ^2 . Therefore, the ratio of particle numbers for the two types of the potential is approximately equal to the ratio of the amplitude squared, which is about 4 at $t = 450 m^{-1}$.

III. RESONANCE IN EXPANDING UNIVERSE

The universe's heating after inflation was achieved due to coupling of the inflaton field ϕ to elementary particle fields. In this process mostly particles with masses smaller than the frequency of the inflaton oscillations were produced. The decay of the inflaton could create both bosons and fermions. The boson production might be strongly enhanced due to excitation of the parametric resonance in the production process [4–8]. Hence bosons were predominantly created initially. Later in the course of thermalization they gave birth to fermions. For a model description of the first stage of this process we assume, as we have done in Sect. II, that the inflaton coupling to a scalar field χ has the form $-g\phi\chi^2/2$, where $g > 0$ is a coupling constant with the dimension of mass. We also assume that the initial value of ϕ is positive. With this choice of the parameters the tachyonic situation can be avoided. Otherwise χ would explosively rise even at inflationary stage. As a result the contribution of χ to the total cosmological energy density would become non-negligible and should be taken into account in the Hubble parameter. This effect may inhibit inflation. These problems will be studied elsewhere. Below we study a simpler situation of the initial stage of heating when the energy density of the produced particles is small in comparison with the energy density of the inflaton, so the back reaction is not of much importance. The effects of the back reaction of particle production on the inflaton evolution are discussed in Sect. IV.

The equation of motion of the Fourier mode of χ with conformal momentum k in the FLRW metric has the form

$$\ddot{\chi} + 3H\dot{\chi} + \left(\frac{k^2}{a^2} + g\phi\right)\chi = 0, \quad (3.1)$$

where $a = a(t)$ is the cosmological scale factor, $H = \dot{a}/a$ is the Hubble parameter, and field χ is taken for simplicity to be massless, $m_\chi = 0$. We assume that the universe is 3D-flat and that the cosmological energy density is dominated by the inflaton field, so H is expressed through ϕ as

$$H = \sqrt{\frac{8\pi}{3}} \frac{\sqrt{\dot{\phi}^2/2 + U(\phi)}}{m_{Pl}}, \quad (3.2)$$

where $U(\phi)$ is the potential of the inflaton, $m_{Pl} \approx 1.2 \cdot 10^{19}$ GeV is the Planck mass, and it is assumed, as usually, that the inflaton field is homogeneous, $\phi = \phi(t)$. Correspondingly the equation of motion for ϕ has the form

$$\ddot{\phi} + 3H\dot{\phi} + U'(\phi) = 0, \quad (3.3)$$

where $U' = dU/d\phi$.

We study here the particle production by ϕ , which evolves in the potential (2.9), described in the previous section, where it has been shown that the particle production in flat space-time is strongly enhanced in comparison with the particle production by ϕ with the potential $U(\phi) = m^2\phi^2/2$. We do the same thing here.

In expanding universe the liquid friction term $3H\dot{\phi}$ in the equation of motion for $\phi(t)$ (3.3) strongly modifies the evolution of ϕ and it is necessary that the resonance should be generated faster than ϕ significantly dropped down.

We find numerical solutions of Eq. (3.3) with the same potentials as above, i.e the harmonic one and $U(\phi)$ presented in Eq. (2.9). As initial conditions we take $\dot{\phi}(0) = 0$ and two different values $\phi(0) = 4.64 m_{Pl}$ and $\phi(0) = 9 m_{Pl}$, which we consider in parallel. The first one, $\phi(0) = 4.64 m_{Pl}$, is equal to the value which we took in the flat universe case (see Fig. 2). However, in this case the initial energies, $U(\phi(0))$, are very different for the two potentials (see Fig. 1). So we consider also the initial condition $\phi(0) = 9 m_{Pl}$ for which the two potentials $U(\phi(0))$ have equal magnitudes. The results of a numerical solution of Eq. (3.3) are shown in Fig. 5.

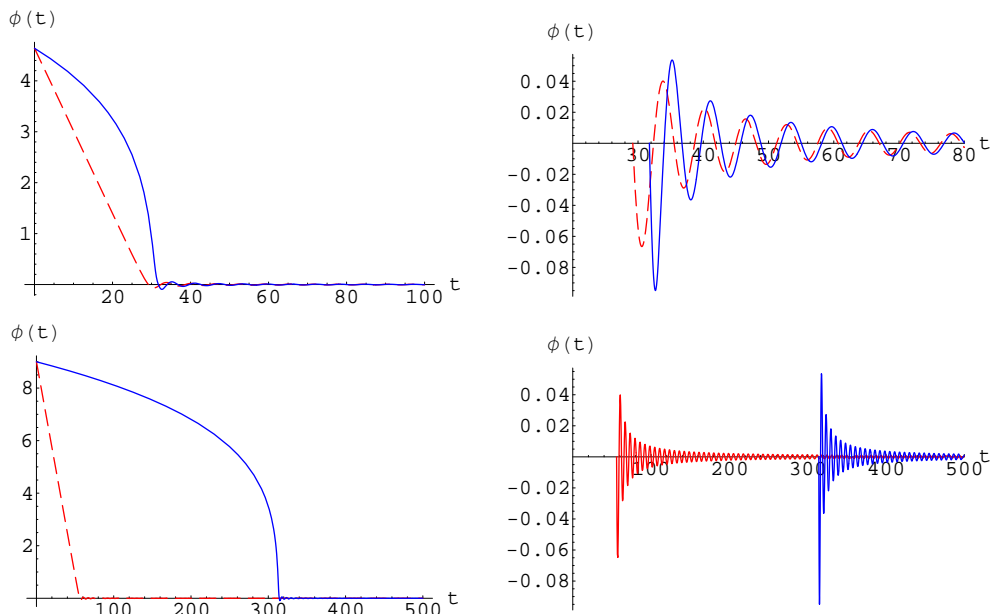


FIG. 5: Inflaton field, $\phi(t)$, in the expanding universe. The time t is measured in units of m^{-1} and field ϕ is measured in units of m_{Pl} . In the *left panels* the evolution $\phi(t)$ is shown starting from $t = 0$. In the *right panels* the oscillations of $\phi(t)$ are presented in more detail, starting from the moment when $\phi = 0$ for the first time. The *upper plots* correspond to the case $\phi(0) = 4.64 m_{Pl}$, and the *lower ones* correspond to the case $\phi(0) = 9 m_{Pl}$.

At the stage of inflation the inflaton field, $\phi(t)$, rolls toward the minimum of potential quite slowly, due to the large "friction" H . For successful inflation one needs the condition $\int_{t_i}^{t_e} H(t)dt > 70$ to be satisfied, where t_i is

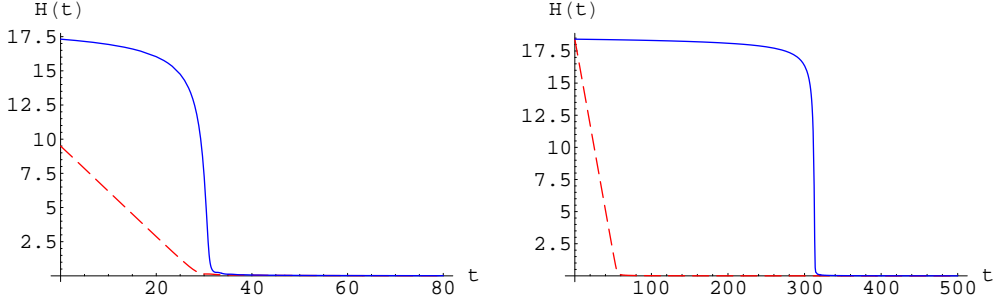


FIG. 6: Hubble parameter $H(t)$. The time t is measured in units of m^{-1} and H is measured in units of m . The left plot corresponds to the case $\phi(0) = 4.64 m_{Pl}$, and the right one corresponds to the case $\phi(0) = 9 m_{Pl}$.

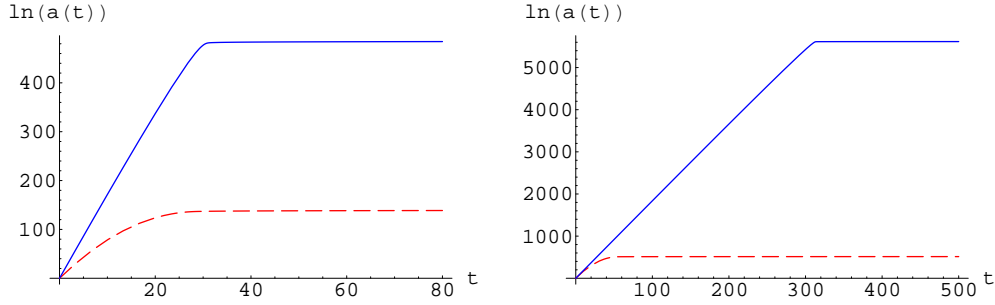


FIG. 7: Logarithm of the scale factor $a(t)$. The time t is measured in units of m^{-1} . The initial value is $a(0) = 1$. The left plot corresponds to $\phi(0) = 4.64 m_{Pl}$, and the right one corresponds to $\phi(0) = 9 m_{Pl}$.

time of the beginning and t_e is the time of the end of inflation. It can easily be seen in Fig. 6 that this condition is fulfilled for both potentials. The scale factor $a(t)$ grows exponentially during inflation (see Fig. 7).

After $\phi(t)$ reaches the minimum of the potential, it does not have enough energy to climb high back because of the energy loss due to the friction, so ϕ starts to oscillate with decreasing amplitude. The moment t_0 , when $\phi(t)$ crosses zero for the first time, can be considered as the end of inflation, the onset of the oscillations, and the universe's heating. The Hubble parameter becomes quite small by this moment, $H \lesssim m$, and continues to decrease, so during the oscillation period one can neglect H in comparison with m .

Equation (3.3) is simplified by the substitution $\phi(t) = \Phi(t)/a^{3/2}(t)$, and in the case of quadratic potential it has a solution $\phi(t) = \phi_0(t) \cos mt$, where $\phi_0(t) \sim a^{-3/2}(t)$. Therefore, Eq. (3.1) turns into

$$\ddot{\chi} + 3H\dot{\chi} + \frac{k^2}{a^2} \left(1 + \frac{g\phi_0 a^2}{k^2} \cos mt \right) \chi = 0, \quad (3.4)$$

which is the Mathieu equation with a friction term, so the condition of parametric resonance [32] is

$$\left| m - \frac{2k}{a} \right| < \sqrt{\left(\frac{g\phi_0 a}{2k} \right)^2 - 9H^2}. \quad (3.5)$$

Let us consider now Eq. (3.1) in the general case. It is convenient to make the substitution $\chi(t) = X(t)/a^{3/2}(t)$, so one obtains

$$\ddot{X} + \left(\frac{k^2}{a^2} + g\phi - \frac{3}{4}H^2 - \frac{3\ddot{a}}{2a} \right) X = 0. \quad (3.6)$$

Equation (3.6) has the form of the free oscillator equation $\ddot{X} + \omega^2 X = 0$ with frequency depending on time. During the oscillations the terms H^2 and \ddot{a}/a are relatively small, so one can take

$$\omega(t) \approx \sqrt{\frac{k^2}{a^2(t)} + g\phi(t)}. \quad (3.7)$$

If we neglect the time dependence of $a(t)$ and take a small g , then ω would be almost constant and the solution of (3.6) is $X(t) \simeq e^{-i\omega t}/\sqrt{2\omega}$, which corresponds to $\chi(t) = e^{-i\omega t}/\sqrt{2\omega V}$, where $V = a^3$ is the comoving volume.

The energy and number of produced particles in a comoving volume are, respectively,

$$E(t) = \frac{\dot{X}^2}{2} + \frac{\omega^2 X^2}{2}, \quad (3.8)$$

$$N(t) = \left(\frac{\dot{X}^2}{2} + \frac{\omega^2 X^2}{2} \right) \frac{1}{\omega}. \quad (3.9)$$

The scale factor $a(t)$ changes much slower with time than $X(t)$, therefore $\dot{\chi} \approx \dot{X}/a^{3/2}$. Thus the energy and the number densities of the produced χ -particles would be

$$\varrho_\chi(t) = \frac{\dot{\chi}^2}{2} + \frac{\omega^2 \chi^2}{2}, \quad (3.10)$$

$$n_\chi(t) = \left(\frac{\dot{\chi}^2}{2} + \frac{\omega^2 \chi^2}{2} \right) \frac{1}{\omega}. \quad (3.11)$$

It is reasonable to impose the initial conditions for the field $\chi(t)$ at the moment t_0 , which is the moment of the onset of the inflaton oscillations. We choose the initial conditions as $\chi(t_0) = 1/\sqrt{2\omega(t_0)V(t_0)}$, $\dot{\chi}(t_0) = \sqrt{\omega(t_0)/2V(t_0)}$, where $V(t_0) = 1$ in units of m^{-3} . These conditions correspond to vanishing initial density of the χ -particles. To avoid the tachyonic situation we put $k/a(t_0) = 50m$ and $g = 5 \cdot 10^{-4}m$, therefore we ensure that ω^2 is always positive during the interesting time interval.

At large time the amplitude of the ϕ oscillation becomes small and the potential (2.9) closely approaches the quadratic one; therefore the condition of parametric resonance (3.5) holds for both potentials. Thus, the resonance occurs in the narrow region when $k/a(t)$ is near $m/2$. In Fig. 8 one can see at what time the resonance occurs for the two potentials of ϕ with the chosen parameters.

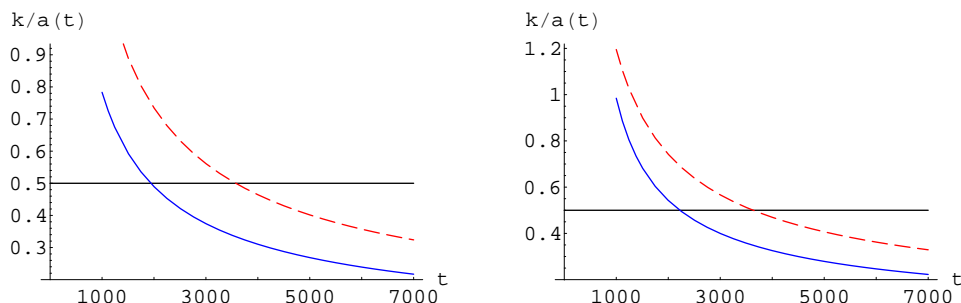


FIG. 8: The physical momentum $k/a(t)$ (in units of m). The black horizontal lines cross the functions $k/a(t)$ in the resonance points, $k/a(t) = m/2$. The left plot corresponds to $\phi(0) = 4.64 m_{Pl}$, and the right one corresponds to $\phi(0) = 9 m_{Pl}$.

We have assumed here that the inflaton field gives the dominant contribution to the cosmological energy density. Therefore, when the energy density of the produced particles, ϱ_χ , becomes comparable to $\varrho_\phi = \dot{\phi}^2/2 + U(\phi)$, the model stops to be self-consistent and we should modify the calculations. The easiest way is to take the energy density of the produced particles at this moment as an ultimate one and to estimate the cosmological heating temperature on the basis of this result. A more precise way is to take into account the back reaction of the produced particles on the damping of the inflaton oscillations and to include the contribution of the created particles into the Hubble parameter. The first simplified approach, which gives a correct order of magnitude estimate of the temperature, is sufficient for our purposes.

Resonance particle production could induce specific features in the primordial spectrum of density perturbations [33], which might be potentially observable. We thank the referee for mentioning this effect. This could be the subject of a separate study.

The energy densities of the produced particles, $\varrho(t)$, for $\phi(0) = 4.64 m_{Pl}$ and $\phi(0) = 9 m_{Pl}$ are presented in Fig. 9. One can see that for the quadratic potential of ϕ the parametric resonance is quite weak and the energy density of particles produced during the resonance is much less than ϱ_ϕ , which is about $3 \cdot 10^3 m^4$ at the moment of the resonance emergence. On the contrary, in the potential (2.9) ϱ_χ increases very quickly and becomes comparable to ϱ_ϕ at $t \approx 2050$ and $t \approx 2330$ in the cases $\phi(0) = 4.64 m_{Pl}$ and $\phi(0) = 9 m_{Pl}$, respectively.

The number densities $n(t)$ calculated according to Eq. (3.11) and the total numbers $N(t) = n(t) \cdot a^3(t)$ of the produced particles are presented in Figs. 10 and 11, respectively. These plots can be understood as follows.

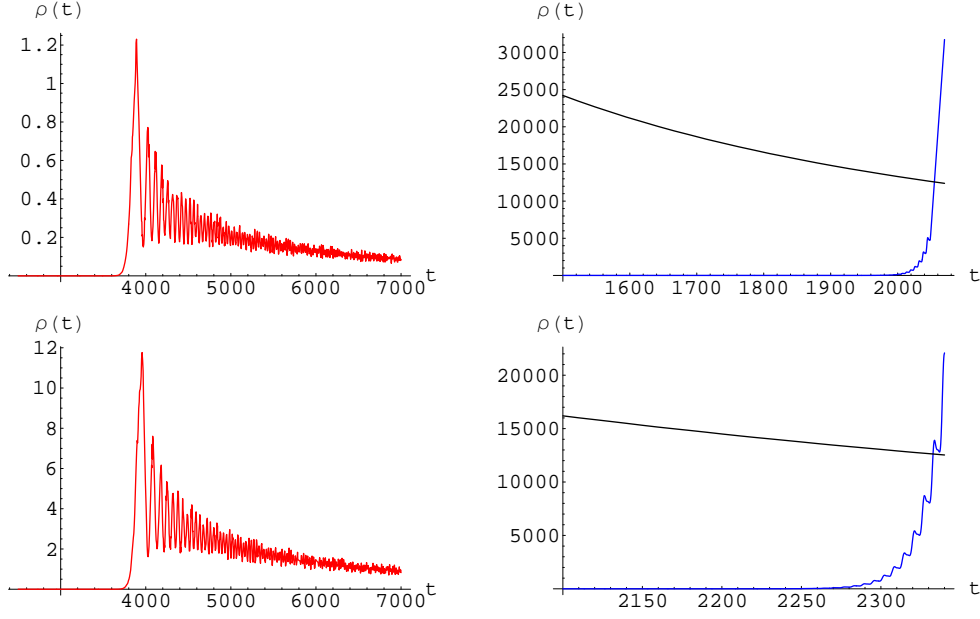


FIG. 9: Energy density, $\varrho(t)$, of the produced particles χ . The time t is measured in units of m^{-1} and $\varrho(t)$ is measured in units of m^4 . The upper plots correspond to $\phi(0) = 4.64 m_{Pl}$, and the lower ones correspond to $\phi(0) = 9 m_{Pl}$. Black lines in the right panels denote ϱ_ϕ

During some time after the beginning of the ϕ oscillations the conditions of parametric resonance are not fulfilled and therefore χ -particles are not produced in a considerable amount. Due to the universe's expansion the term $(k/a(t))^2$ drops down and at some moment the mode with conformal momentum k enters the resonance region. It is manifested as an exponential growth of $\varrho(t)$, $n(t)$, and $N(t)$. Then the resonance conditions stop to be satisfied once again and χ -particle production is almost terminated, so their total number tends to a constant value, while their number density, $n(t)$, decreases as a^{-3} .

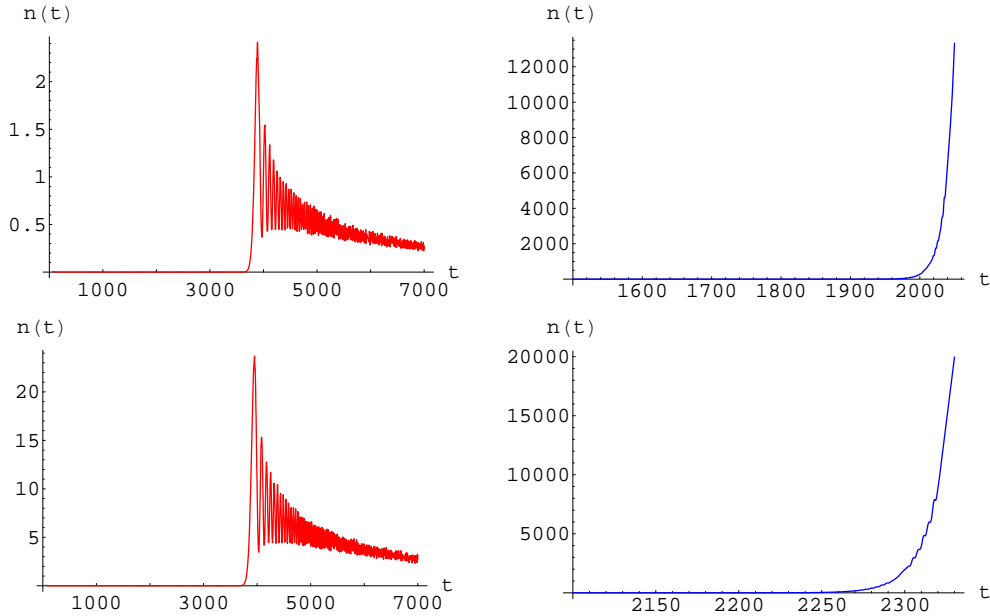


FIG. 10: The number density of the produced χ -particles $n(t)$. The time t is measured in units of m^{-1} and number density is measured in units of m^3 . The upper plots correspond to $\phi(0) = 4.64 m_{Pl}$ and the lower ones correspond to $\phi(0) = 9 m_{Pl}$

Let us suppose that χ -particles reach thermal equilibrium very quickly. The energy density of relativistic

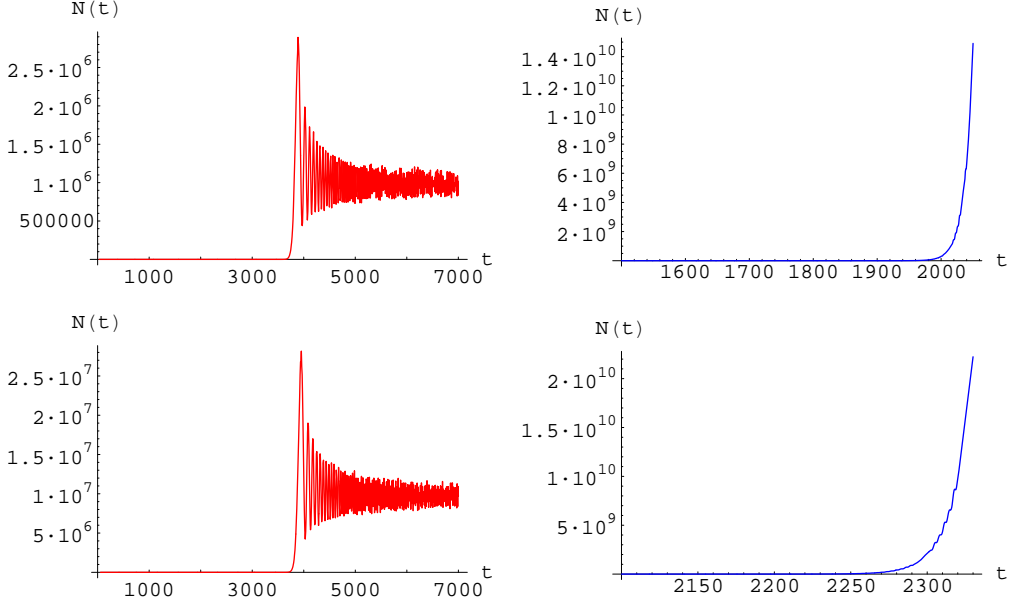


FIG. 11: The total number of produced χ -particles $N(t)$. The time t is measured in units of m^{-1} . The upper plots correspond to $\phi(0) = 4.64 m_{Pl}$, and the lower ones correspond to $\phi(0) = 9 m_{Pl}$

particles is simply related to the temperature:

$$\varrho_\chi = \frac{\pi^2}{30} g_* T^4, \quad (3.12)$$

where $g_* \sim 100$ is the number of relativistic particle species in the thermalized plasma. Therefore, from Fig. 9 in the case of the potential (2.9) one can estimate the temperature by the moment when $\varrho_\chi \sim \varrho_\phi$ as

$$T \sim 5 m \approx 10^{-5} m_{Pl} \sim 10^{14} \text{ GeV}. \quad (3.13)$$

It is instructive to present a qualitative explanation of the obtained results. The initial moment, when the particle production is initiated, is the moment of the onset of the inflaton oscillations. They started when the Hubble friction term in the inflaton equation of motion (3.3) becomes small in comparison with the potential term. It took place when H^2 dropped down below $U''(\phi)$. For the harmonic potential $U_h = m^2 \phi^2/2$ it happened after ϕ reached the boundary value $\phi_h^2 = 3m_{Pl}^2/4\pi$. For our potential (2.9) the oscillation regime is reached at a $\sqrt{6}$ times larger value of ϕ , i.e. $\phi_\lambda^2 = 9m_{Pl}^2/2\pi$. This result is obtained in the case of $\lambda_0 \gg 1$, as has been chosen for the potential (2.9).

Discussing particle production we considered the initial physical momentum of the produced particles $p = k/a_{in} = 50 m$ to avoid a tachyonic excitation, and we took initial conditions for χ corresponding to the vacuum state, i.e. to the state where the real χ -quanta were absent. In the course of the cosmological expansion this momentum evolved down to the resonance value $k/a_{res} = m/2$, both in the cases of harmonic and non-harmonic evolution. Since the Hubble parameter in the modified theory was larger than that in the harmonic case, the former reached the resonance value faster; see Fig. 8. However, this is not directly essential, because in both cases p reaches the resonance value at the same redshift relative to the initial state. An important factor which determines the effectivity of the resonance particle production is the amplitude of the inflaton field at the resonance. Initial amplitudes differed by the factor $\sqrt{6}$, but this is not the end of the story. The amplitude of the "harmonic inflaton" dropped down as $1/a^{3/2}$, while the evolution of the inflaton living in the potential (2.9) is considerably slower. For a purely quartic potential, $\sim \lambda \phi^4$, the amplitude of the inflaton field drops as $1/a$, so it comes to the resonance with the amplitude 10 times larger than the harmonic inflaton.

In the case considered in this paper the modified potential is not purely quartic, moreover, it approaches the harmonic form at small ϕ , when the resonance is excited. Nevertheless, the amplitude of ϕ during the harmonic regime is much larger than in the purely harmonic case, though not by such a large factor.

According to the equations presented in Sect. II the amplitude of χ at the resonance rises as $\sim \exp(g\phi_{res}t/2m)$, where ϕ_{res} is the inflaton amplitude at the resonance. This explains the difference in efficiency of the particle production between the purely harmonic and modified cases.

Finally, it should be noted that production of χ -particles is possible also due to the usual non-resonant decay $\phi \rightarrow \chi\chi$. However, the width Γ of such a decay is very small. Indeed, it follows from Eq. (2.2) that $\Gamma = g^2 m/32\pi$

for neutral massless χ . Therefore, for our choice of the interaction constant, $g = 5 \cdot 10^{-4} m$, the typical time of the decay is $\tau = 1/\Gamma \sim 4 \cdot 10^8 m^{-1}$, which is much longer than the time when the resonance occurs (see Fig. 8). Thus the contribution of the non-resonant decay is negligible.

The inflationary model based on the potential (2.9) is similar to the well-known new inflationary scenario or inflation at a small field ϕ ; see e.g. Ref. [34]. Correspondingly, the slow roll parameters satisfy the condition $\varepsilon \ll |\eta|$. The definition of the parameters and their relation to scalar and tensor perturbations can be found in Refs. [34–37]. The inequality mentioned above means that the amplitude of the gravitational waves in the model considered here must be very small. However, this conclusion is model dependent and is not necessarily true.

IV. BACK REACTION

The tremendous rate of particle production makes its back reaction on the inflaton evolution non-negligible quite soon. The simplest way to estimate this back reaction is to use the density balance condition: the energy loss by the inflaton must be equal to the energy carried away by the produced particles. Such a comparison is done in Fig. 9. Before this moment we can neglect the related decrease of the inflaton amplitude. This is similar to the well-known instant decay approximation, which usually works pretty well. So as regards the order of magnitude we can rely on the results obtained with the neglected back reaction. However, in the instant decay approximation the decay rate remains constant, while the parametric resonance rate is proportional to the amplitude of the inflaton, and so with decreasing ϕ the resonance production drops down and it may happen that the remaining part of the inflaton energy would decay much more slowly. Keeping in mind that usually the inflaton makes nonrelativistic matter, while produced particles are relativistic, we can conclude that the relative contribution of the inflaton to the cosmological energy density would grow and ultimately the dominant part of the (re)heated cosmic plasma would be created by slow (perturbative) inflaton decay. However, this is not a subject of the present work.

For a more accurate treatment of the back reaction we can use the equation of motion of the inflaton with quantum effects induced by particle production in the one loop approximation [38, 39]. The corresponding equation is an integro-differential one, which for the coupling of the produced quantum field χ of the form (2.2) can be written as

$$\begin{aligned} \ddot{\varphi}_c + V'(\varphi_c) - \frac{g^2}{16\pi^2} \varphi_c \ln \frac{t_1}{\epsilon} \\ = \frac{g^2}{16\pi^2} \int_0^{t_1} \frac{d\tau}{\tau} [\varphi_c(t-\tau) - \varphi_c(t)] + \frac{g^2}{16\pi^2} \int_{t_1}^{t-t_{in}} \frac{d\tau}{\tau} \varphi_c(t-\tau) . \end{aligned} \quad (4.1)$$

Here we use the notation φ_c for the classical inflaton field to keep track with Ref. [38]. The integrals in the r.h.s. are ultraviolet finite. The logarithmically infinite contribution in the l.h.s., related to the ultraviolet cut-off $\epsilon \rightarrow 0$, is taken out by the mass renormalization, so with a possible bare mass term in the potential, $V_{m_0} = m_0^2 \varphi_c^2/2$ (here m_0 is the bare mass of φ_c), we obtain

$$\begin{aligned} \ddot{\varphi}_c + m^2(t_1) \varphi_c + [V'(\varphi_c) - V'_m(\varphi_c)] \\ = \frac{g^2}{16\pi^2} \int_0^{t_1} \frac{d\tau}{\tau} [\varphi_c(t-\tau) - \varphi_c(t)] + \frac{g^2}{16\pi^2} \int_{t_1}^{t-t_{in}} \frac{d\tau}{\tau} \varphi_c(t-\tau) , \end{aligned} \quad (4.2)$$

where t_1 is an arbitrary normalization point and the "running" mass is $m^2(t_1) = m^2(t_2) - (g^2/16\pi^2) \ln(t_1/t_2)$. In Eq. (4.2) we explicitly separated the massive part, V_{m_0} , in the potential, so that the term in the square brackets vanishes for the harmonic potential, $V(\varphi) = m_0^2 \varphi^2/2$.

The equation governing the evolution of ϕ_c can be grossly simplified if the particle production goes in the resonant mode. In this case the occupation numbers of the field χ become very large and the field can be treated as a classical one.

We assume, as is usually done in studies of parametric resonance, that χ is a real field. Since the resonance is rather narrow, the universe's expansion can be neglected. Correspondingly, the k -mode of the massless field χ satisfies the following equation of motion:

$$\ddot{\chi}_k(t) + k^2 \chi_k(t) + g \phi_c(t) \chi_k(t) = 0. \quad (4.3)$$

In what follows we omit the subindices k and c .

This equation can be transformed into the integral equation:

$$\chi(t) = \chi_0 - g \int_0^t dt' G_R(t-t') \phi(t') \chi(t'), \quad (4.4)$$

where χ_0 is the initial value of χ , which is assumed for simplicity to be zero (it does not rise exponentially, so can be neglected anyhow) and

$$G_R(t) = \begin{cases} \sin kt/k, & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases} \quad (4.5)$$

is the retarded Green's function of Eq. (4.3).

Making the ansatz $\chi = A \exp(\gamma t) \sin(kt + \alpha)$, and neglecting oscillating terms in the integral (4.4), we find that this ansatz is self-consistent, i.e. χ indeed rises exponentially, if $\phi = \phi_0 \sin mt$, and $k = m/2$, $\alpha = \pi/2$:

$$\chi = A \frac{g\phi_0}{4k\gamma} e^{\gamma t} \cos kt. \quad (4.6)$$

It is assumed here that both A and the amplitude of the inflaton oscillations, ϕ_0 , slowly change with time. For self-consistency one needs to impose the condition $g\phi_0/4k\gamma = 1$. It leads to the canonical expression for γ presented in Eq. (2.7).

Let us turn now to Eq. (2.3). The impact of the χ -particle production originating from the term $-g\chi^2/2$ can be estimated as follows. We substitute expression (4.6) for χ and assume that ϕ evolves as $\phi = \phi_0(t) \sin mt$, where $\phi_0(t)$ is (in comparison with frequency m) a slowly varying function of time. In this way we obtain the equation:

$$2m\dot{\phi}_0 \cos mt = -\frac{1}{2}gA^2 e^{2\gamma t} \cos^2 kt = -\frac{gA^2}{4} e^{2\gamma t} (1 + \cos 2kt). \quad (4.7)$$

The first term in the brackets in the r.h.s. describes the tadpole contribution and should be disregarded. The second term is consistent with the l.h.s. if $k = m/2$, as expected, and ultimately we arrive at the equation

$$\dot{\phi}_0 = -\frac{gA^2}{8m} e^{2\gamma t}. \quad (4.8)$$

As $\dot{\phi}_0$ is always negative, the amplitude of the inflaton oscillation ϕ_0 constantly decreases. Thus, if we neglect variations of γ and take $\gamma = g\phi_{0,res}/2m$, where $\phi_{0,res}$ is the value of ϕ_0 at the beginning of resonance, we obtain the lower limit on the time of the oscillation damping. Indeed, in such a case the amplitude of the inflaton oscillation can easily be found:

$$\phi_0 = \phi_{0,res} - \frac{gA^2}{16m\gamma} e^{2\gamma t}, \quad (4.9)$$

and the characteristic time of the oscillation damping is

$$\tau_d \approx \frac{1}{2\gamma} = \frac{m}{g\phi_{0,res}}. \quad (4.10)$$

According to our calculations $\phi_{0,res} \approx 10^{-4} m_{Pl} = 100m$, and the parameter A can be estimated as $A \sim 0.1m$ (the initial value of χ). Using also $g = 5 \cdot 10^{-4}m$, we find that $\tau_d \sim 20/m$.

In Fig. 12 the energy density of the produced χ -particles is presented, as usually, for $\phi(0) = 4.64 m_{Pl}$ (left panel) and for $\phi(0) = 9 m_{Pl}$ (right panel). The exponential rise is clearly observed with the exponent quite close to the above calculated one.

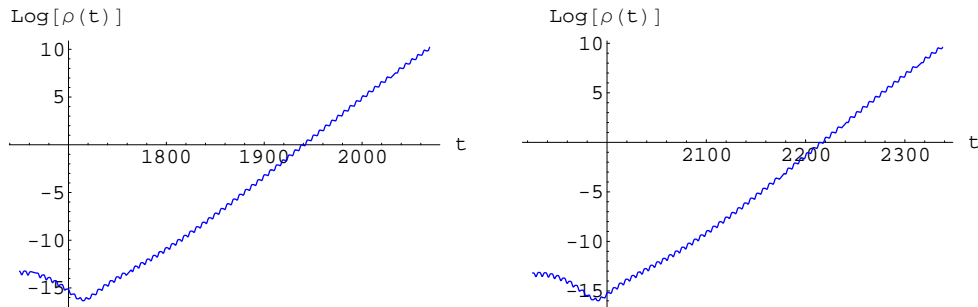


FIG. 12: Energy density of the produced χ -particles for $\phi(0) = 4.64 m_{Pl}$ (left panel) and for $\phi(0) = 9 m_{Pl}$ (right panel) in a normal logarithm scale

The characteristic lapse of time during which ϕ disappears down to zero can be estimated from Eq. (4.9) and is equal to

$$\tau_\phi \sim \frac{1}{2\gamma} \ln \frac{16m\gamma\phi_{0,res}}{gA^2} \sim \frac{m}{g\phi_{0,res}} \ln \frac{8\phi_{0,res}^2}{A^2}. \quad (4.11)$$

With the chosen above parameter values we have $\tau_\phi \approx 300/m$. In reality it would be somewhat longer because with the decreasing amplitude of ϕ the parametric resonance exponent drops down proportionally to ϕ .

V. CONCLUSION

We have shown that even a small modification of the shape of the inflaton oscillations could lead to a significant increase of the probability of particle production by the inflaton and consequently to a higher universe temperature after inflation. The particle (boson) production by the inflaton oscillating in a harmonic potential was compared to the particle production in a toy inflationary model with a flat inflaton potential at infinity. It was found that the parametric resonance in the latter case is excited much more efficiently.

The inflationary model which is considered above is not necessarily realistic. We took it as a simple example to demonstrate the efficiency of the particle production and the heating of the universe. The impact of the energy density of the produced particles on the cosmological expansion may noticeably change the phenomenological properties of the underlying inflationary model.

We skip on purpose a possible tachyonic amplification of χ -excitement to avoid the deviation from the main path of the work on amplification of particle production due to variation of the shape of the inflaton oscillations.

In Ref. [40] a similar study of the impact of the anharmonic corrections to the inflaton oscillations was performed, but the effect is opposite to that advocated in our paper: instead of amplification the anharmonicity leads to a damping of particle production. However, in this paper the form of the inflaton oscillations is different from ours. It shows that the shape of the signal is indeed of crucial importance to the efficiency of the particle production. We thank M. Amin for the indication of Ref. [40]. Recently there have appeared a few more papers [41–43] in which the efficiency of inflationary heating was studied. However, the mechanism considered there is different from ours.

Acknowledgments

AD and AS acknowledge the support of the grant of the Russian Federation government 11.G34.31.0047.

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